Entry-0004

Axiom of Choice implies Zorn's Lemma

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Abstract

We present a proof of Zorn's Lemma based on Lewin's notion of a conforming set. The proof constructs chains recursively using a choice function and shows that all such constructions are forced to agree on their initial segments. Their union therefore forms a single maximal construction, which the choice function paradoxically compels to extend further, yielding a contradiction.

1. Introduction

Theorem 1.1 (Zorn's Lemma)**.** *Let $(X, \leq)$ be a nonempty partially ordered set in which every chain has an upper bound. Then X has a maximal element.*

There are two broad styles of proof of Zorn's lemma. One proceeds via transfinite induction and ordinals; here is the intuition behind such an argument. Let X be a nonempty partially ordered set in which every totally ordered subset has an upper bound. Suppose X has no maximal element. Choose $x_1 \in X$. It is not maximal, so there exists $x_2 > x_1$. But x_2 is not maximal, so there exists $x_3 > x_2$, and so on, leading to

$$x_1 < x_2 < x_3 < \cdots .$$

These elements form a chain, hence there is an element x_ω dominating all of them. But x_ω is not maximal, so there exists $x_{\omega+1}$, and the process continues. Continuing in this way, one is led to an element $x_\alpha \in X$ for every ordinal α. This contradicts the fact that the size of a set is bounded by a fixed ordinal (see the Burali–Forti paradox).

The proof we give below [3] is structural rather than ordinal: it avoids transfinite induction and ordinals, relying instead only on the Axiom of Choice (AC) and elementary extremality arguments. The central idea is to build chains step-by-step using a fixed choice function. The decisive feature is that all such constructions are forced to agree on their initial segments. They therefore assemble into a single coherent maximal construction—one that the choice function itself compels to extend further, yielding a contradiction. In preparation for the proof, Sections 2 and 3 introduce the notation and the key definition used in the argument.

Strategy. We assume that X has no maximal element and fix a choice function selecting a strict upper bound for each chain. We then consider all chains that are generated by repeatedly applying this function. The key step is to show that these constructions are compatible and therefore combine into a single maximal chain, which the choice function forces to extend further, yielding a contradiction.

2. Setup

Assume, for contradiction, that $(X, \leq)$ has no maximal element. By hypothesis, every chain has an upper bound. So, if C is a chain in X, there exists an upper bound u of C. That is, $y \leq u$, for every $y \in C$. Since there is no maximal element, u cannot be a maximal element (in other words, there exists $x \neq u, u \leq x$). So $y < x$ for every $y \in C$. Such an element x will be called a *strict upper bound* of C. Since $(X, \leq)$ has no maximal element, a strict upper bound is not maximal. Thus, associated with every chain is a set of strict upper bounds of the chain.

By the Axiom of Choice, there exists a function that assigns to each chain $C \subseteq X$ a strict upper bound of C. Fix such a function f. Thus

$$f : \{\text{chains in } X\} \to X$$

such that

$$c < f(C) \quad \text{for all } c \in C.$$

3. Conforming sets

For $A \subseteq X$ and $x \in A$, define

$$I(A, x) = \{y \in A : y < x\}.$$

Definition 3.1. A subset $A \subseteq X$ is called conforming if:

1. A is well-ordered,
2. $x = f(I(A, x))$ for every $x \in A$.

Interpretation. The function f is fixed in advance and is not chosen to fit A. Rather, a conforming set is one that conforms to the behavior of the choice function f. A evolves according to f: each element $x \in A$ is exactly the element selected by f from the set of strict upper bounds of $I(A, x)$. Thus each stage of the construction tacks on an element that is strictly greater than the largest element in the previous stage (chain).

As example, consider a chain C which is $x_1 < x_2 < x_3$. In the parent set X, there is a next element, say x_4, of C because every chain has a strict upper bound as seen above. But $f(C)$ may not be x_4. Suppose $f(C) = y$, where y is a strict upper bound. Then $x_1 < x_2 < x_3 < y$ is a new chain and $f(I(A, y)) = y$. A is thus the chain built by f step by step from a chain of $(X, \leq)$. A does not contain every element of the chain in $(X, \leq)$ that A is built from. A has only the elements produced by the choice function f. Thus, not every chain in $(X, \leq)$ is conforming, but some chains are.

Since A is well-ordered, each $I(A, x)$ is a chain in X, so $f(I(A, x))$ is well-defined (in the sense that if $f(I(A, x)) = c_1$ and $f(I(A, x)) = c_2$, then $c_1 = c_2$).

Remark (Construction viewpoint).

Note that $\emptyset$ and singleton sets are chains. Conforming sets may be built inductively: starting with $x_0 = f(\varnothing)$, then

$$x_1 = f(\{x_0\}), \quad x_2 = f(\{x_0, x_1\}), \quad \ldots$$

and continuing transfinitely. A conforming set is precisely a set that arises from such a construction (possibly stopped at some stage).

Condition (2) encodes this entire recursive process: it forces every element of A to appear exactly when dictated by f.

4. Comparability of Conforming Sets (Comparability Lemma)

Guiding idea. Each conforming set is generated step-by-step by the same choice function. Once two such constructions agree on an initial segment, the next element is uniquely determined; hence they cannot diverge.

Lemma 4.1. *If A and B are conforming subsets of X and $A \neq B$, then one of these sets is an initial segment of the other.*

Proof. Recall that A and B are well-ordered and satisfy

$$x = f(I(A, x)) \quad \text{for all } x \in A, \qquad x = f(I(B, x)) \quad \text{for all } x \in B.$$

Define

$$C = \{\, x \in A \cap B : I(A, x) = I(B, x) \,\}.$$

$C \neq \varnothing$, for if a and b are, respectively, the least elements of A and B, then, since $I(A, a) = I(B, b) = \varnothing$, $a = f(\varnothing) = b$, and hence $a = b \in C$.

Claim. C is an initial segment of both A and B.

Let $x \in C$ and let $u \in A$ with $u < x$. We show that $u \in C$, proving that C is an initial segment of A.

Since $u < x$ and $x \in C$, we have

$$u \in I(A, x) = I(B, x),$$

so $u \in B$.

We now prove

$$I(A, u) = I(B, u).$$

Let $v \in I(A, u)$. Then $v < u < x$, so $v \in I(A, x) = I(B, x)$, hence $v \in B$ and $v < u$, so $v \in I(B, u)$. Thus

$$I(A, u) \subseteq I(B, u).$$

The reverse inclusion follows by symmetry, so

$$I(A, u) = I(B, u).$$

Thus $u \in C$.

This shows that every element of A below x lies in C, so C is an initial segment of A. The same argument applies to B. Thus C is an initial segment of both A and B, proving the claim.

If both $A \setminus C$ and $B \setminus C$ are nonempty, we compare the first points at which the two constructions A and B diverge. Define

$$a = \min(A \setminus C), \qquad b = \min(B \setminus C).$$

Claim. $I(A, a) = I(B, b) = C$.

We prove

$$I(A, a) = C.$$

First, let $u \in I(A, a)$. Then $u \in A$ and $u < a$. Since $a = \min(A \setminus C)$, there is no element of $A \setminus C$ smaller than a. Thus every element of A below a must lie in C. So $u \in C$. Hence

$$I(A, a) \subseteq C.$$

Conversely, let $u \in C$. We show $u < a$.

- If $u = a$, this contradicts $a \notin C$.
- If $a < u$, then since $u \in C$ and C is an initial segment of A, every element of A below u lies in C. But $a \in A$ and $a < u$, so $a \in C$, contradiction.

Thus $u < a$, so $u \in I(A, a)$, and hence

$$C \subseteq I(A, a).$$

Therefore

$$I(A, a) = C.$$

Similarly,

$$I(B, b) = C.$$

This proves the claim.

Since A and B are conforming,

$$a = f(I(A,a)) = f(C), \qquad b = f(I(B,b)) = f(C).$$

Thus

$$a = b.$$

Let $c = a = b$. Then $c \in A$ and $c \in B$.

Also,

$$I(A,c) = I(B,c) = C,$$

Since $c \in A \cap B$ and $I(A,c) = I(B,c) = C$, by definition of C we have $c \in C$. But $c = a \in A \setminus C$, contradiction.

Therefore at least one of $A \setminus C$ or $B \setminus C$ is empty. Thus either $A = C$ or $B = C$; so one is an initial segment of the other.

□

5. The union U of all conforming subsets of X

Lemma 5.1. *The union U of all conforming subsets of X is conforming.*

Proof. We first show that U is a chain. Let $x, y \in U$. There are conforming sets A_x and A_y such that $x \in A_x$ and $y \in A_y$. The Comparability Lemma implies that one of these sets is a subset of the other, so without loss of generality suppose that $A_x \subseteq A_y$. Then $x, y \in A_y$, and A_y is a chain, so either $x \le y$, or $y \le x$.

Now we show that U is well-ordered by $\le$. The key point is that all conforming sets are comparable by initial segments, so their union behaves as a single well-ordered construction rather than a disjoint aggregation. Suppose that $\emptyset \ne S \subseteq U$; we want to show that S has a least element under $\le$. Fix $s \in S$. Choose a conforming set A with $s \in A$. A is well-ordered by $\le$, so let $a = \min\{x \in A \cap S : x \le s\}$; we claim that $a = \min S$. Let $x \in S$; if $x \in A$, then certainly $a \le x$, so suppose that $x \in S \setminus A$. There is some conforming A_x such that $x \in A_x$. Since $x \notin A$, we have $A_x \not\subset A$. By the Comparability Lemma, one of A_x and A is an initial segment of the other. Since A_x is not a subset of A, it follows that A is an initial segment of A_x. Since also $x \in A_x \setminus A$ and $a \in A$, we have $a < x$ in this case as well; hence $a = \min S$.

It remains to show that if $x \in U$, then $x = f(I(U,x))$. Let $x \in U$; there is a conforming set A_x such that $x \in A_x$. By definition $x = f(I(A_x,x))$, so we're done if we can show that $f(I(U,x)) = f(I(A_x,x))$. It suffices to show that $I(U,x) = I(A_x,x)$. Since $A_x \subset U$, $I(A_x,x) \subseteq I(U,x)$. So we need only show that $I(U,x) \subseteq I(A_x,x)$. So suppose $y \in I(U,x)$. That is, $y \in U$ and $y < x$. Since U is the union of all conforming subsets of X, there is a conforming set A_y such that $y \in A_y$. Then, A_x and A_y being conforming sets, one of them is an initial segment of the other. If A_y is the initial segment of A_x, then $y \in A_y \subseteq A_x$. Thus $y \in A_x$ and $y < x$. So $y \in I(A_x,x)$. On the other hand, if A_x is an initial segment of A_y, then $A_x \subseteq A_y$. Since $y \in A_y$ and $y < x$, and A_x is an initial segment of A_y, every element of A_y below x must lie in A_x. Thus $y \in A_x$, proving that $I(U,x) \subseteq I(A_x,x)$. □

6. Proof of Zorn's Lemma

Proof. Suppose X has no maximal element. Since U is a conforming subset of X, there exists $f(U)$ so that

$$u < f(U) \quad \text{for all } u \in U.$$

So $f(U) \notin U$. Thus the construction U, which aggregates all possible conforming constructions, is maximal among such constructions but is still not maximal in X: the choice function forces it to extend further. Let $U^* = U \cup \{f(U)\}$.

U^* is conforming:

U^* is well-ordered since U is well-ordered and $f(U)$ lies above all elements of U.

For $x \in U$, $x = f(I(U, x)) = f(I(U^*, x))$.

For $f(U)$:

$$I(U^*, f(U)) = U,$$

so

$$f(U) = f(I(U^*, f(U))).$$

Now U is the union of all conforming subsets of X. Since U^* is conforming, it must be contained in U. Thus $f(U) \in U$, contradicting that $f(U)$ is strictly greater than every element of U. □

Reader Takeaway.
A conforming set grows by repeatedly applying the fixed choice function, so each stage is forced by what came before. Because all such constructions agree on initial segments, their union U forms a single maximal construction. But the choice function then forces U itself to extend, yielding a contradiction—so a maximal element must exist.

7. Conceptual Summary

A conforming set is a transfinite construction governed by the choice function:

$$x_\alpha = f(\{x_\beta : \beta < \alpha\}).$$

The Comparability Lemma shows that all such constructions are compatible (they agree on initial segments), so their union forms a maximal transfinite construction.

8. Toward the Equivalence of AC, Zorn's Lemma, and Well-Ordering

The argument above is more than a proof of Zorn's Lemma from the Axiom of Choice: it reveals a general structural principle underlying several fundamental results in set theory.

Remark. This proof may be viewed as replacing ordinal recursion with a compatibility principle: instead of building a single transfinite sequence indexed by ordinals, one considers all possible constructions generated by the choice function and shows that they are forced to agree on their initial segments, thereby assembling into a single coherent object.

The key feature is that a global object is built by extending partial constructions step-by-step, with the Axiom of Choice ensuring that each extension is possible.

From this perspective, the Axiom of Choice provides the mechanism that allows such a construction to continue at every stage. The Comparability Lemma shows that all such constructions agree on their initial segments, so they fit together into a single structure. The union U may therefore be viewed as the result of a maximal transfinite construction.

Zorn's Lemma can thus be understood as a principle asserting that such recursively built structures must reach a maximal stage. The contradiction arises because the choice function forces any such maximal construction to extend further.

This viewpoint forms one side of a well-known equivalence:

$$\text{Axiom of Choice} \iff \text{Zorn's Lemma} \iff \text{Well-Ordering Theorem.}$$

In the reverse direction, Zorn's Lemma can be used to show that every set admits a well-ordering, and from this one can recover the Axiom of Choice. The essential idea is again the same: one organizes partial constructions (such as partial well-orderings) into a poset and uses maximality to obtain a global object.

Thus these three principles are not isolated results but different expressions of a single underlying idea:

Global structure emerges from consistent local choices.

References

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