Entry-0006

Cauchy's Integral Formula

Area: Complex Analysis • **Subarea:** Complex Integration

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Abstract

Cauchy's integral formula expresses the value of an analytic function at a point in terms of its values on a surrounding curve. The proof proceeds by isolating the contribution of a single interior point: an auxiliary function is constructed so that its contour integral vanishes, either by boundedness near the point or by a sharper decay condition. This reveals that the entire integral is governed by the local behavior of the integrand near that point. Consequently, analytic functions are completely determined by their boundary values, possess derivatives of all orders, and admit power series expansions.

1. Introduction

An important consequence of Cauchy's Theorem is the representation of an analytic function as a line integral. The representation, known as Cauchy's Integral formula, immediately shows that a differentiable complex function is in fact infinitely differentiable. The formula also helps in the study of the local properties of an analytic function.

We call an open subset D of the complex plane a region if any two points of D can be joined by a polygonal path comprising points of D. A function f that is differentiable at every point in a region D is indicated as $f \in H(D)$. A simple closed path will be abbreviated as *scp*. We denote the interior and exterior regions of a scp Γ as Γ^i and Γ^e, respectively.

We begin by extending Cauchy's theorem from a single scp to configurations involving several such paths. This extension will allow us to isolate points inside a curve and ultimately lead to the integral formula.

2. Cauchy's Theorem for a sum of simple closed paths

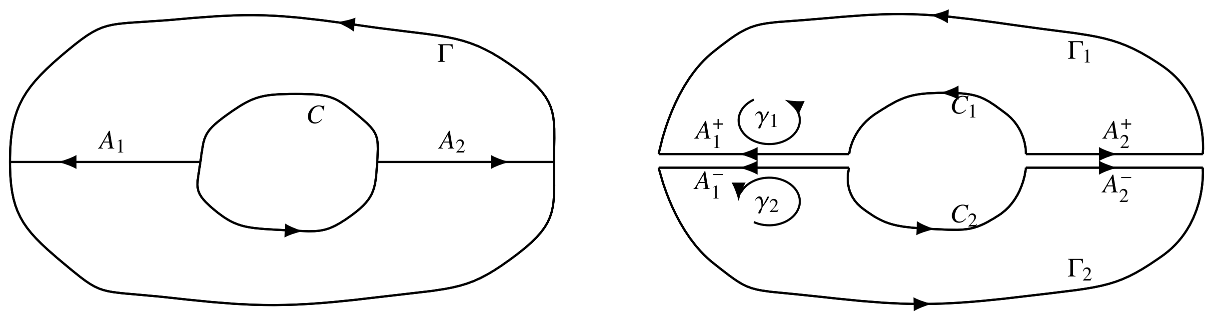
A configuration comprising an scp Γ and one or more scp's Γ_j with each Γ_j inside Γ but outside the other Γ_k (that is, $\Gamma_j \subset \Gamma_k^e \cap \Gamma^i$) can be represented as a single scp that is a sum of all the scp's, as would become clear in the proof of the theorem below.

Theorem 2.1. *Let C and Γ be positively oriented scp's with $C \subset \Gamma^i$. Let $f \in H(D)$ and D contain the closure of the region $\Gamma^i \cap C^e$. Then*

$$\int_\Gamma f(z)dz = \int_C f(z)dz.$$

Idea of the proof: We connect the curves by auxiliary arcs so as to decompose the region between them into two simply connected regions, each bounded by a simple closed path lying in D. Applying Cauchy's theorem to each gives the result.

Proof. Figure 1(a) shows oriented auxiliary arcs A_1 and A_2 joining the curves in $\Gamma^i \cap C^e$. Figure 1(b) shows Figure 1(a) split for illustration only. Let $\gamma_1 = \Gamma_1 - A_1^+ - C_1 + A_2^+$ and $\gamma_2 = \Gamma_2 - A_2^- - C_2 + A_1^-$. Each curve γ_k is obtained by combining portions of Γ and C with auxiliary arcs that lie entirely in the region $\Gamma_i \cap C^e \subset D$. Hence every point enclosed by γ_k lies in D, so Cauchy's theorem applies: $\int_{\gamma_k} f(z)dz = 0$, for $k = 1, 2$. Since $\gamma_1 + \gamma_2 = \Gamma_1 + \Gamma_2 - C_1 - C_2 = \Gamma - C$, we obtain $\int_{\Gamma - C} f(z)dz = 0$, as desired.



(a) (b)

Figure 2.1: (a) A simple closed path C inside Γ joined by arcs A_1 and A_2. (b) Decomposition into two positively oriented simple closed paths γ_1 and γ_2.

(After the slit is introduced, the resulting curves are no longer smooth at the endpoints of the slit. These points are corners, and the curves are therefore only piecewise smooth. This level of regularity is sufficient for the validity of contour integration.) □

Theorem 2.2 (Cauchy's theorem on sum of scp's)**.** *Let* $\Gamma_1, \dots, \Gamma_n$ *and* Γ *be positively oriented scp's such that* $\Gamma_j \subset \Gamma_k^e$ *if* $j \neq k$, *and* $\Gamma_j \subset \Gamma^i$ *for* $j = 1, \dots, n$. *Then* $E = \Gamma^i \cap \Gamma_1^e \cap \cdots \cap \Gamma_n^e$ *is a region. If* D *is a region containing the closure of* E *and if* $f \in H(D)$, *then*

$$\int_\Gamma f(z)dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z)dz.$$

(The proof proceeds by induction on n, repeatedly applying Theorem 2.1 to remove one inner curve at a time. We omit the details.)

3. An extension of Cauchy's theorem

Here we consider a multi-curve version of Cauchy's theorem. This allows us to remove small neighborhoods of finitely many points inside Γ, reducing the study of integrals to arbitrarily small contours around those points.

Lemma 3.1. *Let* Γ *be a simple closed path such that* $\Gamma \cup \Gamma^i$ *is contained in a region* $D \subset \mathbb{C}$. *Let* $z_1, \dots, z_n$ *be points in* Γ^i *and let* D^* *be the region obtained by deleting these points from* D. *Let*

$f \in H(D^*)$ and suppose that there exists $\delta > 0$ such that f is bounded in a deleted neighborhood $N'_\delta(z_j)$ for each z_j. Then $\int_\Gamma f(z)dz = 0$.

Idea of the proof. We remove small circles around each deleted point and apply Theorem 2.2. The integral over Γ is then expressed as a sum of integrals over these small circles, which can be made arbitrarily small.

Proof. Let $\epsilon > 0$. If $|f(z)| < M$ in each $N'_\delta(z_j)$, consider circles $\Gamma_j \subset \Gamma^i$ centered at each z_j and of radius r, where r is chosen less than δ so small that no two Γ_j meet, and so that $r < \frac{\epsilon}{2\pi M n}$. From Theorem 2.2, $\int_\Gamma f(z)dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z)dz$. From Theorem 4.1 in [3], $|\int_C f(z)dz| \leq \int_C |f|ds \leq (M.\text{length of } C)$ for a smooth curve C. Hence we have

$$\left|\int_\Gamma f(z)dz\right| \leq \sum_{j}^{n} \left|\int_{\Gamma_j} f(z)dz\right| \leq n.M.(2\pi r) < \epsilon.$$

It follows that $\int_\Gamma f(z)dz = 0$. □

Thus, finitely many isolated points at which f is bounded do not contribute to the value of the integral.

Thus, the study of contour integrals is reduced to understanding the contribution from arbitrarily small contours surrounding individual points. The next step is therefore to analyze what happens near a single point inside Γ.

The preceding lemma uses boundedness near the deleted points. A related and useful condition is that $(z - z_j)f(z) \to 0$, which ensures that the contribution from small circles vanishes.

Lemma 3.2. *Let D be a region, and let Γ be a positively oriented simple closed path such that $\Gamma \cup \Gamma_i \subset D$. Let $z_1, \ldots, z_n$ be points in Γ_i, and set*

$$D^* = D \setminus \{z_1, \ldots, z_n\}.$$

Suppose $f \in H(D^*)$, and assume that for each $j = 1, \ldots, n$,

$$\lim_{z \to z_j} (z - z_j)f(z) = 0.$$

Then

$$\int_\Gamma f(z)\, dz = 0.$$

Idea of the proof. We remove small circles around each deleted point and apply the multi-curve version of Cauchy's theorem. The hypothesis implies that $f(z)$ grows more slowly than $1/|z - z_j|$, so the contribution from each small circle tends to zero.

Proof. Let $\varepsilon > 0$. For each $j = 1, \ldots, n$, the condition

$$\lim_{z \to z_j} (z - z_j)f(z) = 0$$

implies that there exists $\delta_j > 0$ such that

$$|(z - z_j)f(z)| < \frac{\varepsilon}{2\pi n} \quad \text{whenever } 0 < |z - z_j| < \delta_j.$$

Choose $r > 0$ small enough that $r < \delta_j$ for all j, and such that the circles

$$\Gamma_j = \{z : |z - z_j| = r\}$$

are pairwise disjoint and lie entirely inside Γ_i.

By Theorem 2.2,

$$\int_\Gamma f(z)\, dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z)\, dz.$$

On each Γ_j, we have $|z - z_j| = r$, so

$$|f(z)| = \frac{|(z - z_j)f(z)|}{|z - z_j|} < \frac{\varepsilon}{2\pi n r}.$$

Hence, using the estimate for contour integrals,

$$\left|\int_{\Gamma_j} f(z)\, dz\right| \leq \max_{\Gamma_j} |f| \cdot \text{length}(\Gamma_j) \leq \frac{\varepsilon}{2\pi n r} \cdot (2\pi r) = \frac{\varepsilon}{n}.$$

Summing over j,

$$\left|\int_\Gamma f(z)\, dz\right| \leq \sum_{j=1}^{n} \left|\int_{\Gamma_j} f(z)\, dz\right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\int_\Gamma f(z)\, dz = 0.$$

4. Index of a point relative to a closed curve

For a point that is not on a closed curve, index is a notion making precise the number of times the closed curve winds around the point. For piecewise smooth curves, the definition of index is based on the following lemma.

Lemma 4.1. *If a smooth closed curve Γ does not pass through the point a , then $\int_\Gamma \frac{dz}{z-a}$ is a multiple of $2\pi i$.*

Proof. Let $\Gamma : z(t), \alpha \leq t \leq \beta$ be a smooth closed curve. That is, $z'(t)$ exists and $z'(t)$ is continuous. Now

$$\int_\Gamma \frac{dz}{z - a} = \int_\Gamma \frac{z'(t)}{z(t) - a} dt.$$

Let

$$g(t) = \int_{\alpha}^{t} \frac{z'(s)}{z(s) - a} ds.$$

Since $\frac{z'(s)}{z(s)-a}$ is continuous, g is differentiable. For, if $\frac{z'(s)}{z(s)-a} = f(s)$, then $\frac{g(t+\delta)-g(t)}{\delta} = \frac{\int_t^{t+\delta} f(s)ds}{\delta}$ so that taking $\lim_{\delta \to 0}$ gives $g'(t) = f(t)$ due to continuity of f. This is the Fundamental theorem of calculus.

But the derivative of $e^{-g(t)}(z(t) - a)$ vanishes. Therefore $e^{-g(t)}(z(t) - a)$ is a constant K for $\alpha \le t \le \beta$.

Since Γ is a closed curve, $z(\alpha) = z(\beta)$. Hence

$$Ke^{g(\alpha)} = z(\alpha) - a = z(\beta) - a = Ke^{g(\beta)}.$$

Since $z(\alpha) \neq a$, we have $K \neq 0$, and therefore

$$e^{g(\alpha)} = e^{g(\beta)}.$$

It follows that there exists $n \in \mathbb{Z}$ such that

$$g(\beta) = g(\alpha) + 2\pi in.$$

Since $g(\alpha) = 0$, we obtain

$$g(\beta) = 2\pi in.$$

□

This result shows that the integral measures the net change in the argument of $z(t) - a$ along the curve.

Remark 4.2. If Γ is a piecewise smooth closed curve, write Γ as a finite union of smooth curves

$$\Gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n,$$

where each γ_i is defined on $[a_i, b_i]$ and $b_i = a_{i+1}$.

On each segment γ_i, the argument above applies and shows that

$$e^{-g(t)}(z(t) - a)$$

is constant. Since g is continuous on the whole interval and the endpoints match

$$z(b_i) = z(a_{i+1}),$$

these constants must agree across adjacent segments.

Thus the same constant holds along the entire curve, and the conclusion

$$g(\beta) = 2\pi in$$

follows exactly as in the smooth case.

Definition 4.3. In view of $\int_\Gamma \frac{dz}{z-a} = 2\pi in$, the index of a relative to Γ is defined as

$$n(\Gamma, a) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z - a}.$$

The integer $n(\Gamma, a)$ counts, with sign, the number of times the curve Γ winds around the point a. It depends only on the continuous deformation class of the curve avoiding a.

5. Cauchy's Integral

Let $f \in H(D)$, and let Γ be a simple closed curve in D. Let a lie inside Γ. Define

$$F(z) = \frac{f(z) - f(a)}{z - a}, \qquad z \in D \setminus \{a\}.$$

Since f is analytic in D, the function F is analytic on $D \setminus \{a\}$.

Moreover,

$$(z - a)F(z) = f(z) - f(a).$$

Since f is continuous at a, we have

$$\lim_{z \to a} (z - a)F(z) = \lim_{z \to a} \big(f(z) - f(a)\big) = 0.$$

But $F \in H(D \setminus \{a\})$ and the condition

$$\lim_{z \to a} (z - a)F(z) = 0$$

show that F satisfies the hypothesis of Lemma 3.2 (with the single omitted point a). It follows that

$$\int_\Gamma F(z)\, dz = 0.$$

Hence

$$\int_\Gamma \frac{f(z) - f(a)}{z - a}\, dz = 0,$$

so

$$\int_\Gamma \frac{f(z)}{z - a}\, dz - f(a) \int_\Gamma \frac{1}{z - a}\, dz = 0.$$

Therefore

$$\int_\Gamma \frac{f(z)}{z - a}\, dz = f(a) \int_\Gamma \frac{1}{z - a}\, dz.$$

By Lemma 4.1, the integral $\int_\Gamma \frac{dz}{z-a}$ depends only on how the curve winds around a, and is given by

$$\int_\Gamma \frac{1}{z - a}\, dz = 2\pi i n.$$

Thus the entire integral is determined by the value of f at the point a and the number of times the curve Γ winds around a. If Γ winds once positively around a, so that $n(\Gamma, a) = 1$, this yields

$$\boxed{f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - a}\, dz.}$$

This shows that the contribution of the point a to the integral is completely local, and is detected through the winding of the curve around a.

Key idea. The auxiliary function

$$F(z) = \frac{f(z) - f(a)}{z - a}$$

may fail to be defined at a, but its singularity there is mild enough that Lemma 3.2 applies: the quantity $(z - a)F(z)$ tends to 0.

Remark 5.1 (Alternative proof). There is a second route to Cauchy's integral formula using the removable singularity result (Lemma 3.1).

Define

$$F(z) = \frac{f(z) - f(a)}{z - a}, \qquad z \in D \setminus \{a\}.$$

Then $F \in H(D \setminus \{a\})$, and

$$\lim_{z \to a} F(z) = f'(a).$$

To verify boundedness near a, note that since the limit exists, there is $\delta > 0$ such that

$$\left| \frac{f(z) - f(a)}{z - a} - f'(a) \right| < 1 \quad \text{whenever } 0 < |z - a| < \delta.$$

Hence, for $0 < |z - a| < \delta$,

$$\left| \frac{f(z) - f(a)}{z - a} \right| \leq \left| \frac{f(z) - f(a)}{z - a} - f'(a) \right| + |f'(a)| < 1 + |f'(a)|.$$

Thus F is bounded in a deleted neighborhood of a.

Since $F \in H(D \setminus \{a\})$ and is bounded near a, Lemma 3.1 applies, and we conclude that

$$\int_\Gamma F(z)\, dz = 0.$$

The remainder of the argument proceeds as before, yielding

$$f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - a}\, dz.$$

Comparison. This proof uses boundedness near the singularity to conclude that its contribution to the integral vanishes. In contrast, the proof using Lemma 3.2 replaces boundedness by the sharper condition $(z - a)F(z) \to 0$, avoiding even this estimate. The latter approach is therefore more economical.

6. Consequences

The integral formula allows us to analyze the behavior of analytic functions at points inside the curve Γ. One of its most remarkable consequences is that a function that is once complex differentiable is in fact infinitely differentiable.

To make this precise, we study a family of integral operators associated with a curve.

Lemma 6.1. *Suppose that $\phi(\zeta)$ is continuous on the arc γ. Then the function*

$$F_n(z) = \int_\gamma \frac{\phi(\zeta) d\zeta}{(\zeta - z)^n}$$

is analytic in each region determined by γ, and its derivative is

$$F_n'(z) = nF_{n+1}(z).$$

Proof omitted; the result follows by differentiating under the integral sign. See [2] for details. Now, applying the Integral formula with $\phi(\zeta) = f(\zeta)$ and $F_1(z) = f(z)$, we have that $f(z)$ is differentiable:

$$f'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$$

and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}$$

Thus a function that is once complex differentiable admits derivatives of all orders. While this conclusion is obtained via an integral representation over a curve Γ enclosing the point, the differentiability is a local property of the function itself; the curve serves only as a device through which this structure is revealed.

From this we obtain a partial converse of Cauchy's theorem. Cauchy's theorem asserts that analytic functions have vanishing integrals over closed curves. Morera's theorem shows that the converse holds under an additional hypothesis: if a function is continuous and its integral over every closed curve vanishes, then it is analytic. The continuity assumption is essential; without it, the vanishing of all such integrals does not in general guarantee analyticity.

Theorem 6.2 (Morera's theorem)**.** *Let f be continuous in the region D and satisfy $\int_\Gamma f(z)dz = 0$ for every scp in D. Then $f \in H(D)$.*

Indeed, the vanishing of integrals over closed curves suggests that the integral of f depends only on the endpoints of a path. This allows one to define a local primitive, from which differentiability follows.

A consequence of infinite differentiability of analytic functions is the following,

Theorem 6.3 (Liouville's theorem)**.** *A function which is analytic and bounded in the whole plane must reduce to a constant.*

And Loiuville's theorem implies the fundamental theorem of algebra: a polynomial P(z) of degree >0 has a root.

7. Closing perspective

Above all, Cauchy's integral formula reveals a remarkable principle: the value of an analytic function inside a curve is completely determined by its values on the boundary. There is no hidden interior freedom.

At the same time, the proof isolates a deeper phenomenon. The behavior of the function

$$\frac{f(z)}{z - a}$$

in an arbitrarily small neighborhood of the point a, enclosed by the curve, determines the value of the integral. In particular, the contribution of a single point is detected through a contour integral.

Thus local behavior and global structure are inseparably linked: what happens near a point controls what is measured along the curve. This principle has far-reaching consequences: analytic functions possess derivatives of all orders and admit power series expansions, both determined by their boundary values.

This viewpoint leads naturally to the classification of isolated singularities and the theory of residues.

References

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[2] L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw–Hill, 1979.

[3] ProofZoom Entry-0003, *Line integrals on rectifiable curves*, ProofZoom (2026).

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