

Entry-0009

Ordinal Numbers and the Shape of Order

Part I: Foundations and Representation

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Abstract

Cardinal numbers measure size. Ordinal numbers measure order. This essay explains why mathematics needs ordinal numbers, why well-ordered sets have order types, and why the von Neumann construction gives a canonical representative for each well-ordered order type. The guiding idea is simple but profound: an ordinal is a number that records position in a well-ordering.

In Part I, we develop the structural foundations of von Neumann ordinals, establish their rigid linearity under Trichotomy, and prove the fundamental Representation Theorem: every well-ordered set is structurally identical to a unique ordinal.

1 Two Questions: How Many, and In What Order?

The natural numbers first enter our lives as answers to the question:

How many?

Three apples. Five books. Ten students. In this use, the number tells us the size of a collection.

But mathematics eventually asks another question:

In what order?

Who comes first? Who comes next? What comes after every finite stage? Is there a position after all the natural numbers?

Cardinal numbers answer the first question. They measure size. Ordinal numbers answer the second question. They measure position and order.

Thus the distinction is:

Cardinals measure size; ordinals measure order.

For finite collections, this distinction is easy to miss. A line of eight students has eight students, and its positions are first, second, third, and so on up to eighth. Size and order seem to move together. But infinity separates them.

Consider the ordered list $0, 1, 2, 3, \dots$. This sequence has no last element. If we append a new element at the limit, obtaining $0, 1, 2, 3, \dots, \omega$, the set remains countable. Its cardinality hasn't changed, but its geometry has: the first order has no terminal element; the second one does. Cardinality measures structural volume; ordinals measure structural shape.

2 Order Type: Forgetting Names, Remembering Positions

Suppose we have two ordered lists:

$$a_0, a_1, a_2, a_3, \dots \quad \text{and} \quad b_0, b_1, b_2, b_3, \dots$$

The names of the elements are different. But the order-pattern is the same: first element, second element, third element, and so on without end. What matters is not the names a_n or b_n , but their relative positions.

Mathematically, two ordered sets have the same order type if there is a bijection between them that preserves order. Such a bijection is called an **order-isomorphism**.

Thus, two well-ordered sets A and B have the same order type if there exists a bijection $f : A \rightarrow B$ such that for all $x, y \in A$:

$$x <_A y \iff f(x) <_B f(y).$$

In words: f sends earlier elements to earlier elements, later elements to later elements, and faithfully preserves the entire order-pattern. This matches our intuitive goal: an ordinal should name the order type of a well-ordered set. To make this precise, we must rely on a remarkable property of well-orderings: their total rigidity.

Proposition 1 (Rigidity of Well-Orderings). *Let $(S, <)$ and $(S', <')$ be well-ordered sets. If there is an order-isomorphism from S to S' , then it is unique.*

Proof. Suppose $\phi, \psi : S \rightarrow S'$ are two distinct order-isomorphisms. We shall prove that $\phi = \psi$.

If $S = \emptyset$, the claim holds vacuously. Assume $S \neq \emptyset$. Since $\phi \neq \psi$, the set of elements where the mappings disagree, defined as:

$$D = \{x \in S : \phi(x) \neq \psi(x)\}$$

is nonempty. Because S is well-ordered, D must contain a unique least element, say a . By choice of a , we have $\phi(a) \neq \psi(a)$, while $\phi(x) = \psi(x)$ holds for all $x < a$.

Since S' is totally ordered, trichotomy implies that either $\psi(a) <' \phi(a)$ or $\phi(a) <' \psi(a)$. Because the cases are symmetric, we assume without loss of generality that $\psi(a) <' \phi(a)$.

We will show that $\psi(a)$ cannot lie in the range of ϕ . By analyzing the placement of an arbitrary element $x \in S$ relative to a , the strict order-preserving properties of our isomorphisms yield that $\phi(x) \neq \psi(a)$ for all x .

► **PZoom: Prove that $\psi(a)$ escapes the image of ϕ**

So $\psi(a) \notin \text{im}(\phi)$. This directly contradicts the fact that ϕ is a surjective order-isomorphism onto S' . Thus, our initial assumption that a point of disagreement exists must be false, proving that $\phi = \psi$. $\square$

This structural rigidity means that if two well-ordered sets share an order type, there is exactly one way to match their elements. The first element must map to the first, the second to the second, and so on. This rigidity is precisely what allows us to look for a unique, canonical representative for each order type.

3 The Tempting Definition and Why It Fails

A natural idea is this: call two well-ordered sets equivalent if they are order-isomorphic, and then define an ordinal number as the equivalence class of all well-ordered sets sharing a given order type. For example, every eight-element well-ordered set has the same finite order type, so perhaps the ordinal 8 should be the collection of all well-ordered sets with eight elements.

This captures the right intuition, but it encounters a catastrophic set-theoretic barrier: the collection of all well-ordered sets of a given order type is too large to be a set. It is a proper class.

Even in the simplest case, the difficulty appears. Consider the order type of a one-element well-ordered set. Every singleton set has this order type:

$$\{a\}, \quad \{b\}, \quad \{c\}, \quad \dots$$

But there is no set of all singleton sets, because that would imply the existence of a set of all possible mathematical objects. The collection overflows the boundaries of Zermelo-Fraenkel set theory.

Nor can we solve this by simply declaring, “Choose one representative from each equivalence class.” This is exactly what the von Neumann construction achieves.

4 von Neumann’s Idea: Each Position Is the Set of Earlier Positions

The decisive idea is to build each position out of the positions that come before it. The first position has nothing before it. So define:

$$0 = \emptyset.$$

The second position has only 0 before it. So define:

$$1 = \{0\}.$$

The third position has 0 and 1 before it. So define:

$$2 = \{0, 1\}.$$

The fourth position has 0, 1, and 2 before it. So define:

$$3 = \{0, 1, 2\}.$$

Thus, in general, a finite position is defined by the collection of its predecessors:

$$n = \{0, 1, 2, \dots, n-1\}.$$

The fundamental paradigm is now clear:

An ordinal is the set of all earlier ordinals.

This same paradigm naturally transcends the finite. The first infinite position is simply the collection of all finite numbers:

$$\omega = \{0, 1, 2, 3, \dots\}.$$

Because every finite ordinal has an immediate successor, ω contains no last element. We can then comfortably define the next position immediately following it:

$$\omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, \dots, \omega\}.$$

This construction maps order directly onto set membership ($\in$). Earlier positions become elements of later positions.

5 Definition of an Ordinal

We now formalize this construction. A set α is called **transitive** if every element of α is also a subset of α :

$$x \in \alpha \implies x \subseteq \alpha.$$

Equivalently, if $x \in y \in \alpha$, then $x \in \alpha$.

Definition 1 (von Neumann ordinal). A **von Neumann ordinal** is a set α satisfying two properties:

- (Or1) α is transitive.
- (Or2) The membership relation $\in$ well-orders α .

For any two elements $\beta, \gamma \in \alpha$, we define their order relation by membership:

$$\beta < \gamma \iff \beta \in \gamma.$$

An ordinal is therefore not merely an arbitrary well-ordered set; it is a well-ordered set where the strict order relation is explicitly chosen to be set membership. This makes the conceptual slogan perfectly precise:

$$\boxed{\alpha = \{\beta : \beta < \alpha\}.$$

6 Roadmap of Part I

In this first part of our essay, we systematically develop the foundational anatomy of ordinals, demonstrating how they organize themselves into a coherent structural universe, culminating in the Representation Theorem.

1. **Members of ordinals are ordinals (Section 7).** We show that ordinals are internally coherent; opening an ordinal reveals only smaller ordinals.
2. **Trichotomy: Any two ordinals are comparable (Section 8).** We establish that the class of ordinals is perfectly linear and non-branching.
3. **Transitive sets of ordinals are ordinals (Section 9).** We provide a practical test for ordinality, demonstrating that a gapless set of ordinals is itself an ordinal.
4. **The Representation Theorem (Section 10).** We prove that every abstract well-ordered set is order-isomorphic to exactly one canonical von Neumann ordinal.

With these foundations secure, Part II will introduce ordinal arithmetic, map out the transfinite structural hierarchy ($\omega \cdot 2, \omega^2, \omega^\omega$), navigate the Burali-Forti paradox, and construct the first uncountable ordinal (ω_1).

7 Members of Ordinals Are Ordinals

We have defined an ordinal as a transitive set well-ordered by membership. The first structural fact is that when we look inside an ordinal, we find earlier ordinals.

Proposition 2. *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.*

This is exactly what we should expect from the von Neumann picture. Since an ordinal is the set of all earlier ordinals, every element of an ordinal ought to be an ordinal itself. To establish this formally, we must verify that β satisfies both requirements of the definition:

- (i) β is transitive.
- (ii) The membership relation $\in$ well-orders β .

Proof. Let α be an ordinal and let $\beta \in \alpha$.

Step 1: The membership relation well-orders β .

Since α is an ordinal, the relation $\in$ well-orders α . Because α is transitive, $\beta \in \alpha$ implies that $\beta \subseteq \alpha$.

Since β is a subset of α , the membership relation on β is precisely the restriction of the well-ordering $\in$ from α . Because any subset of a well-ordered set inherits well-ordering under a restricted relation, $\in$ naturally well-orders β .

Step 2: β is transitive.

To show that β is transitive, we must establish that any element belonging to β is also a subset of β . By leveraging the transitivity of the parent ordinal α and the properties of its strict total membership ordering, the relation $x \in y \in \beta$ forces $x \in \beta$.

► **PZoom:** Verify that β is a transitive set

Conclusion:

Since β is transitive and well-ordered by $\in$, it is a von Neumann ordinal. □

Every member of an ordinal is an ordinal.

8 Trichotomy: Any Two Ordinals Are Comparable

We have shown that the members of an ordinal are ordinals. The next result shows something stronger: all ordinals lie on a single, non-branching line. Given two ordinals, one must be a member of the other, or they must be equal. To prove this, we establish three foundational lemmas.

Lemma 1 (Structure of Transitive Subsets). *A transitive subset of an ordinal is an initial segment of that ordinal.*

Recall that for a well-ordered set α and an element $\xi \in \alpha$, the initial segment determined by ξ is defined as $I_\alpha(\xi) = \{x \in \alpha : x < \xi\}$. Since the ordering on ordinals is defined by set membership, this becomes $I_\alpha(\xi) = \{x \in \alpha : x \in \xi\} = \alpha \cap \xi$. Because ordinals are transitive sets, $\alpha \cap \xi = \xi$. Thus, the initial segments of an ordinal α are precisely its elements.

Proof. Let α be an ordinal and $\gamma \subseteq \alpha$ be a transitive subset such that $\gamma \neq \alpha$.

1. **Find the least element missing from γ :** Since γ is a proper subset of α , the relative complement $\alpha \setminus \gamma$ is non-empty. Because α is well-ordered by $\in$, the set $\alpha \setminus \gamma$ must contain a unique least element. Let us call this element ξ_0 .

2. **Show $\gamma \subseteq \xi_0$:**

Let $x \in \gamma$. Since x and ξ_0 are both elements of the ordinal α , they must be comparable under membership.

- If $x = \xi_0$, then $\xi_0 \in \gamma$, contradicting the fact that $\xi_0 \in \alpha \setminus \gamma$.
- If $\xi_0 \in x$, then because γ is a transitive set and $x \in \gamma$, it follows that $\xi_0 \in \gamma$, yielding the same contradiction.
- Therefore, by elimination, it must be that $x \in \xi_0$. Thus, $\gamma \subseteq \xi_0$.

3. **Show $\xi_0 \subseteq \gamma$:**

Let $y \in \xi_0$. Since α is transitive and $\xi_0 \in \alpha$, we know $y \in \alpha$. Because $y \in \xi_0$, we have $y < \xi_0$. Since ξ_0 was defined as the least element of $\alpha \setminus \gamma$, any element in α strictly smaller than ξ_0 cannot be in $\alpha \setminus \gamma$. Therefore, y must belong to γ , meaning $\xi_0 \subseteq \gamma$.

Since $\gamma \subseteq \xi_0$ and $\xi_0 \subseteq \gamma$, we have $\gamma = \xi_0$. Because $\xi_0 \in \alpha$, γ is an element of α , meaning it is precisely an initial segment. $\square$

Lemma 2 (Intersection of Ordinals). *For any two ordinals α and β , their intersection $\gamma = \alpha \cap \beta$ is also an ordinal, and γ is an initial segment of both α and β .*

Proof. If $x \in \gamma$, then $x \in \alpha$ and $x \in \beta$. Since α and β are transitive, any $y \in x$ must belong to both α and β , so $y \in \gamma$, proving γ is transitive. Furthermore, since $\gamma \subseteq \alpha$ and α is well-ordered by $\in$, γ inherits this well-ordering. Thus, γ is an ordinal. Since γ is a transitive subset of both α and β , Lemma 1 implies it is an initial segment of both. $\square$

Lemma 3 (The Proper Subset Property). *For any ordinals μ and ν , if $\mu \subseteq \nu$ and $\mu \neq \nu$, then $\mu \in \nu$.*

Proof. Since $\mu \subsetneq \nu$, the set $\nu \setminus \mu$ is non-empty. Since ν is well-ordered by $\in$, $\nu \setminus \mu$ has a least element, say ξ_0 . Because ξ_0 is the least element of $\nu \setminus \mu$, any element $x \in \nu$ that is strictly smaller than ξ_0 must belong to μ .

Conversely, because μ is an ordinal, it is a transitive set. Since $\mu \subsetneq \nu$, μ is a transitive proper subset of the ordinal ν . By Lemma 1, μ must be an initial segment of ν .

Since the initial segments of an ordinal are precisely its elements, and ξ_0 is the first missing element, the set of elements smaller than ξ_0 is exactly μ :

$$\mu = \{x \in \nu : x < \xi_0\} = \xi_0$$

Since $\xi_0 \in \nu$, it follows immediately that $\mu \in \nu$. $\square$

Theorem 1 (Trichotomy for Ordinals). *For any two ordinals α and β , exactly one of the following holds:*

$$\alpha \in \beta, \quad \beta \in \alpha, \quad \alpha = \beta.$$

Proof. Mutual Exclusivity: No two conditions can hold simultaneously. This follows from the Axiom of Regularity, which forbids $\in$ -cycles. If $\alpha = \beta$, then $\alpha \in \beta \implies \alpha \in \alpha$, a contradiction. If $\alpha \in \beta$ and $\beta \in \alpha$, then by transitivity of β , we get $\alpha \in \alpha$, which is impossible.

Existence: Let α and β be ordinals, and let $\gamma = \alpha \cap \beta$. By Lemma 2, γ is an ordinal and a subset of both.

If $\gamma = \alpha$, then $\alpha \subseteq \beta$. By Lemma 3, this means either $\alpha = \beta$ or $\alpha \in \beta$. If $\gamma = \beta$, then $\beta \subseteq \alpha$, meaning either $\beta = \alpha$ or $\beta \in \alpha$.

If $\gamma = \beta$, then $\beta \subseteq \alpha$, meaning either $\beta = \alpha$ or $\beta \in \alpha$.

Assume for contradiction that $\gamma \neq \alpha$ and $\gamma \neq \beta$. By applying our structural lemmas to this joint proper subset condition, we force a membership configuration that directly violates the Axiom of Regularity.

► **PZoom:** Exhaust the contradiction where γ is a proper subset of both

□

9 Transitive Sets of Ordinals

The preceding properties reveal a unique feature of ordinals: if one ordinal is a proper subset of another, it is promoted from a subset to an actual member of the larger ordinal. We can now resolve the broader structural question: When is an arbitrary set of ordinals itself an ordinal? The answer is remarkably elegant: the only missing requirement is transitivity.

Proposition 3. *Let X be a set of ordinals. Then X is an ordinal if and only if X is transitive.*

Proof. If X is an ordinal, then X is transitive by definition. This establishes the forward direction.

Conversely, suppose X is a transitive set of ordinals. To prove that X is an ordinal, we must verify that it satisfies both core requirements:

- (i) X is transitive.
- (ii) The membership relation $\in$ well-orders X .

Since transitivity is given by our hypothesis, it remains only to show that $\in$ forms a valid well-ordering on X . A relation is a well-ordering if it constitutes a total order and ensures every nonempty subset contains a unique least element.

Step 1: The relation $\in$ totally orders X .

Let $\alpha, \beta \in X$. Because X is a set of ordinals, both α and β are ordinals. By the Trichotomy Theorem, exactly one of the following conditions must hold:

$$\alpha \in \beta, \quad \beta \in \alpha, \quad \alpha = \beta.$$

Thus, any two distinct elements of X are strictly comparable under membership. It follows that $\in$ totally orders X .

Step 2: Every nonempty subset of X possesses a least element.

Let $S \subseteq X$ be a nonempty subset. We must demonstrate that S contains an element which precedes all other members under the relation $\in$.

By the Axiom of Regularity, because S is a nonempty set, there must exist an element $\alpha \in S$ that is disjoint from S under membership:

$$\alpha \cap S = \emptyset$$

We claim that α is the required least element of S .

To verify this, we select any distinct element $\gamma \in S$. By utilizing Trichotomy on these two ordinals, assuming $\gamma \in \alpha$ forces an intersection with S that directly violates our Regularity condition, leaving $\alpha \in \gamma$ as the only logical outcome.

► **PZoom: Exhaust the Regularity contradiction to prove α is minimal**

Conclusion:

Every nonempty subset of X has a least element under $\in$. Consequently, $\in$ well-orders X . Since X is both transitive and well-ordered by membership, it is a von Neumann ordinal. $\square$

This proposition provides a powerful, practical criterion for recognizing ordinals. A collection of ordinals behaves as an ordinal precisely when it is structurally complete—containing all the preceding elements demanded by its internal components. That structural completeness is exactly what transitivity guarantees.

For instance, consider the ordinal $3 = \{0, 1, 2\}$. It forms an ordinal because it is a transitive set. Conversely, the subset $\{1\} \subseteq 3$ fails to be an ordinal because it lacks transitivity:

$$0 \in 1 \in \{1\}, \quad \text{but} \quad 0 \notin \{1\}.$$

This highlights a foundational paradigm:

A set of ordinals is an ordinal exactly when it contains no gaps.

It cannot contain isolated values; it must systematically encompass both its members and the exhaustive collection of their predecessors.

10 The Representation Theorem for Well-Orderings

We have now established three foundational structural facts: members of ordinals are ordinals, the class of ordinals is perfectly linear under trichotomy, and a transitive set of ordinals is itself an ordinal. These properties lay the necessary groundwork for the central structural theorem of ordinal theory.

The Representation Theorem asserts that von Neumann ordinals are not merely examples of well-ordered sets; they are the canonical representatives of all possible well-ordered shapes.

Theorem 2 (Representation Theorem for Well-Orderings). *Every well-ordered set is order-isomorphic to a unique ordinal.*

To prove this, we require a crucial lemma establishing that well-ordered sets are intrinsically rigid and cannot collapse into their own history.

Lemma 4. *No well-ordered set is order-isomorphic to one of its proper initial segments.*

Proof. (Proof of Lemma) Let $(W, <)$ be a well-ordered set, and let $I_b = \{w \in W : w < b\}$ be the proper initial segment of W determined by some $b \in W$. Suppose for contradiction that there exists an order-isomorphism $h : W \rightarrow I_b$.

Since $h(b) \in I_b$, it follows by definition that $h(b) < b$. This means the set of elements displaced downwards by h , defined as:

$$D = \{w \in W : h(w) < w\}$$

is non-empty, as $b \in D$. Since W is well-ordered, D must contain a unique least element; let us call it d .

Because $d \in D$, we have $h(d) < d$. Since h is an order-isomorphism, it strictly preserves the order relation. Applying h to this inequality yields:

$$h(h(d)) < h(d)$$

This inequality implies that the element $h(d)$ satisfies the defining condition of D . Thus, $h(d) \in D$. However, because $h(d) < d$, this directly contradicts our selection of d as the *least* element of D .

By contradiction, no such order-isomorphism can exist. □

With this lemma in hand, we can cleanly prove the uniqueness and existence of our ordinal representatives.

Proof of the Representation Theorem. We divide the proof into two parts: Uniqueness and Existence.

Part 1: Uniqueness

Suppose a well-ordered set A is order-isomorphic to two distinct ordinals, α and β . By composing the isomorphisms, it follows that α and β must be order-isomorphic to each other.

By the Trichotomy Theorem, exactly one of the following holds: $\alpha \in \beta$, $\beta \in \alpha$, or $\alpha = \beta$.

- If $\alpha \in \beta$, then because β is transitive, α is a proper subset of β consisting precisely of the elements preceding α . Thus, α is a proper initial segment of β . By our Lemma, β cannot be order-isomorphic to one of its proper initial segments, ruling out $\alpha \in \beta$.
- By an identical symmetric argument, $\beta \in \alpha$ is impossible.

Therefore, we are forced to conclude that $\alpha = \beta$, confirming that the representing ordinal is unique.

Part 2: Existence

Let $(A, <)$ be an arbitrary well-ordered set. For each $x \in A$, let $A_x = \{y \in A : y < x\}$ denote the initial segment of A determined by x .

Let us say an element $x \in A$ is *good* if its initial segment A_x is order-isomorphic to some ordinal. By our uniqueness proof, if such an ordinal exists, it is unique; we denote it by α_x . We will show by well-ordered induction on A that every element of A is good.

Fix $x \in A$, and assume the inductive hypothesis: every $y < x$ is good. We must prove that x is good. Define the set β as the collection of all ordinal representatives generated below x :

$$\beta = \{\alpha_y : y < x\}$$

We claim that β is an ordinal. Since β is a set of ordinals, by our proposition on transitive sets of ordinals, we need only show that β is transitive.

Let $\gamma \in \beta$. By definition, $\gamma = \alpha_y$ for some specific $y < x$. To prove transitivity, we must show that $\gamma \subseteq \beta$. Let $\delta \in \gamma$. Because $\gamma = \alpha_y$, we have $\delta \in \alpha_y$.

Recall that A_y is order-isomorphic to α_y via some unique mapping. Under this isomorphism, the element $\delta \in \alpha_y$ must correspond to an element $z \in A_y$, meaning $z < y$. Furthermore, this isomorphism maps the initial segment of A_y below z —which is exactly A_z —onto the initial segment of α_y below δ . Since α_y is an ordinal, its initial segment below δ is exactly δ . This establishes an order-isomorphism between A_z and δ .

Because α_z is the unique ordinal isomorphic to A_z , we have $\delta = \alpha_z$. Since $z < y < x$, it follows that $\alpha_z \in \beta$, which means $\delta \in \beta$. Thus, $\gamma \subseteq \beta$, proving that β is a transitive set of ordinals, and therefore an ordinal itself.

Now, define the assignment mapping:

$$F : A_x \rightarrow \beta \quad \text{where} \quad F(y) = \alpha_y$$

By construction, F maps A_x surjectively onto β . Furthermore, if $y < z$ in A_x , then A_y is a proper initial segment of A_z . This forces their unique ordinal representatives to satisfy $\alpha_y \in \alpha_z$. Thus, F preserves and reflects strict order, making it an order-isomorphism $A_x \cong \beta$. Consequently, x is good.

By the principle of well-ordered induction, every element in A is good. Every element $x \in A$ is uniquely order-isomorphic to a unique von Neumann ordinal α_x . Finally, we define the full ordinal representative for the entire set A :

$$\alpha = \{\alpha_x : x \in A\}$$

Finally, we define the full ordinal representative for the entire set A : $\alpha = \{\alpha_x : x \in A\}$. Replicating the exact structural argument used for β , α is a transitive set of ordinals and is therefore an ordinal.

► **PZoom: Verify that α is a transitive set of ordinals**

The map $G : A \rightarrow \alpha$ given by $G(x) = \alpha_x$ yields a total order-isomorphism from A onto α . Thus, A is order-isomorphic to a unique ordinal. $\square$

This theorem represents the precise set-theoretic realization of the intuitive concept of an order type. While an arbitrary well-ordered set may be composed of a chaotic variety of mathematical objects—whether they are points, functions, or abstract matrices—its underlying order-pattern is captured perfectly by exactly one canonical von Neumann ordinal. Ordinals do not merely exemplify well-orderings; they serve as their universal names.

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