Conformality

1 Ray

Denote the complex plane by $\mathbb{C}$ and let z denote a typical complex number. Let $a, d \in \mathbb{C}$ with $d \neq 0$. A *ray* (half-line) with *endpoint* a is a subset of $\mathbb{C}$:

$$R(a,d) = \{a + dt, t \geq 0, d \neq 0\} \tag{1}$$

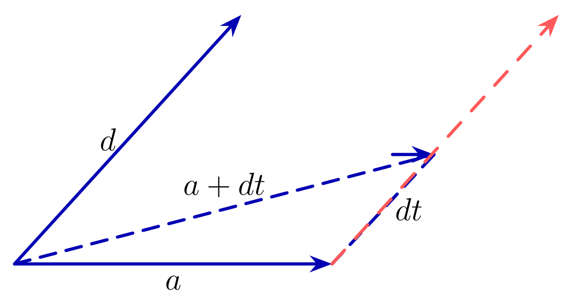

Figure 1: A ray with endpoint a and direction d.

The *opposite ray* to (1) is $\{a - dt, t \geq 0, d \neq 0\}$. It is clear that if $b \in \mathbb{C}$ and $b \neq a$, then

$$R(a,b) = \{a + (b-a)t, t \geq 0\} \tag{2}$$

is a ray with endpoint a which contains b. (Take $t = 1$.)

2 Angle

Angle ψ from $R(a,b)$ to $R(a,c)$ is seen geometrically (Fig. 2) to be $arg(c - a) - arg(b - a)$. When angles are regarded as lying in the range $(-\pi, +\pi]$, the above difference (angle) is between $-\pi$ and $+\pi$. And $(-\pi, +\pi]$ represents anticlockwise rotation which will be assumed as the positive rotation.

Angle θ in $(-\pi, +\pi]$ is said to be the *principal argument* of θ, denoted $Arg\ \theta$. Since $z = |z|e^{i\theta}$, we have $arg\ z = \theta$ due to the vector representing z being inclined at angle θ to the positive real axis of $\mathbb{C}$. Thus the measure M of the angle from b to c at the endpoint of a is

$$M(R(a,b), R(a,c)) = (Arg(c-a) - Arg(b-a))\ modulo\ 2\pi.$$

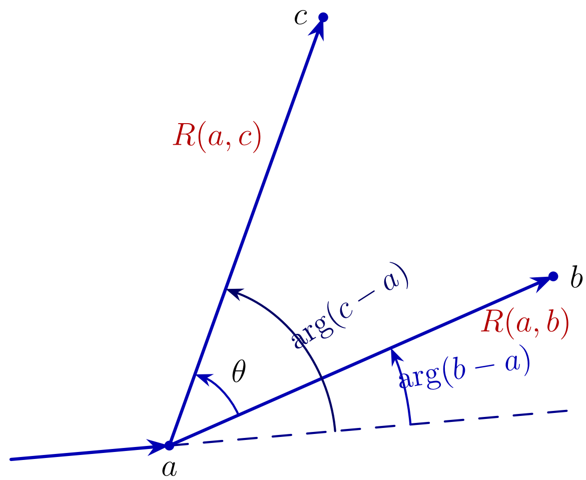

Figure 2: Angles measured at a: the rays $R(a, b)$ and $R(a, c)$, the dashed reference ray, and the angles $\arg(b - a)$, $\arg(c - a)$, and $\theta = Arg(c - a) - Arg(b - a)$.

3 Tangent to a curve

Let $C: \ z(t), \ \alpha \leq \beta$, be a curve, let $t_0 \in (\alpha, \beta)$, and suppose $z'(t_0)$ exists and $z'(t_0) \neq 0$. Let $z_0 = z(t_0)$. Suppose $h > 0$ and consider the ray (Fig. 3)

$$R(z_0, z(t_0 + h)) = \{z : z = z_0 + s(z(t_0 + h) - z_0)/h, 0 \leq s\}.$$

Since

$$lim_{h \to 0^+} \frac{z(t_0 + h) - z_0}{h} = z'(t_0) \neq 0,$$

this ray 'approaches' the ray

$$T(z_0) = \{z : \ z = z_0 + sz'(t_0), \ s \geq 0\}$$

as $h \to 0^+$.

Similarly, if $h > 0$, the ray

$$R(z_0, z(t_0 - h)) = \{z : \ z = z_0 + s(z(t_0 - h) - z_0)/h, 0 \leq s\}$$

approaches the ray opposite to $T(z_0)$. Of these two rays, we call $T(z_0)$ *the tangent ray* because its direction agrees with the direction of travel of the curve C. Thus a curve with $z'(t_0) \neq 0$ possesses a definite local direction, which will serve as the geometric basis for defining angles between curves.

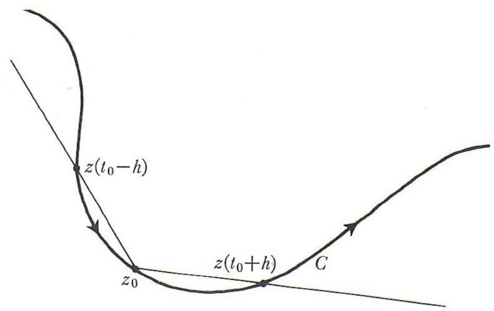

Figure 3: Secants on a point of a curve

Remark 1. *The condition $z'(t_0) \neq 0$ guarantees a well-defined first-order tangent direction. However, a tangent may still exist at a point where $z'(t_0) = 0$, in which case the direction is determined by higher-order terms.*

4 Invariance under regular reparametrization

Theorem 2. *The non-vanishing of $z'(t_0)$ is preserved under regular reparametrizations of the curve.*

Proof. Let $\tau = \phi(t)$ be a C^1 change of parameter with $\phi'(t_0) \neq 0$, and set $\tau_0 = \phi(t_0)$. Define the reparametrized curve

$$\tilde{z}(\tau) = z(\phi^{-1}(\tau)).$$

By the chain rule,

$$\tilde{z}'(\tau_0) = z'(t_0)\,(\phi^{-1})'(\tau_0).$$

Since $\phi'(t_0) \neq 0$, the inverse function theorem (see the Remark following Theorem 6) gives

$$(\phi^{-1})'(\tau_0) = \frac{1}{\phi'(t_0)}$$

Hence

$$\tilde{z}'(\tau_0) = \frac{z'(t_0)}{\phi'(t_0)}.$$

Since $\phi : I \subset \mathbb{R} \to J \subset \mathbb{R}$ is a C^1 change of parameter, it is a real-valued function of a real variable, and hence $\phi'(t_0) \in \mathbb{R}$. Since a regular reparametrization means that the change of parameter has nonzero derivative, we have $\phi'(t_0) \neq 0$, and therefore $\phi'(t_0) \in \mathbb{R} \setminus \{0\}$.

In particular,

$$\tilde{z}'(\tau_0) \neq 0 \Longleftrightarrow z'(t_0) \neq 0,$$

since $\phi'(t_0) \neq 0$.

□

Consequently, $\tilde{z}'(\tau_0)$ is a nonzero real scalar multiple of $z'(t_0)$, so the tangent direction is unchanged, and the tangent ray depends only on the geometric curve and not on the choice of admissible parameter. Hence the existence of a well-defined first-order tangent direction is intrinsic to the curve itself.

5 Angle between curves

Definition 3. *Let C_1 and C_2 be (oriented) path segments which intersect at z_0 (Fig. 4). Let $T_1(z_0)$ and $T_2(z_0)$ be the tangent rays to C_1 and C_2, respectively. The angle from C_1 to C_2 is defined to be the oriented angle from the tangent ray $T_1(z_0)$ to the tangent ray $T_2(z_0)$.*

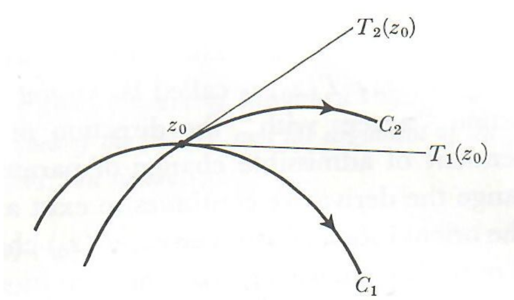

Figure 4: Two paths (curves), with $z(t)$ having derivatives that are continuous (meaning, the curve traced by $z'(t)$ is not disconnected)

The parametric representations $z_1(t)$ and $z_2(t)$ of C_1 and C_2 are arranged so that $z_0 = z_1(t_0) = z_2(t_0)$. Then

$$\begin{aligned}\angle(C_1, C_2) &= \angle(T_1, T_2)\\ &= arg\ T_2(z_0) - arg\ T_1(z_0)\\ &= arg(z_2(t_0) + sz_2'(t_0)) - arg(z_1(t_0) + rz_1'(t_0))\\ &= arg(z_0 + sz_2'(t_0)) - arg(z_0 + rz_1'(t_0))\end{aligned}$$

Since T_1 and T_2 are rays at the endpoint of z_0, the angle between them is
$arg\big(z_0 + sz_2'(t_0)\big) - arg(z_0 + rz_1'(t_0)) = arg(sz_2'(t_0)) - arg(rz_1'(t_0))$.
And since s and r are scalars,
$arg(sz_2'(t_0)) - arg(rz_1'(t_0)) = arg(z_2'(t_0)) - arg(z_1'(t_0))$, which is $Arg\frac{z_2'(t_0)}{z_1'(t_0)}$.

6 Conformality

The term 'conformal' refers to the property that, for a map from U to V, $U, V \subseteq \mathbb{C}$, the angle between the curves in U is the same as the angle between their image curves in V. For example, under the mapping $w = e^z$, vertical and horizontal lines map onto circles and radial rays orthogonal to the circles. This preservation of orthogonality is a manifestation of conformality.

Let $C : z(t), \alpha$, be a regular (meaning, $z'(t)$ exists for all $t \in [\alpha, \beta]$) path segment in $\mathbb{C}$ and $z_0 = z(t_0)$, where $\alpha < t_0 < \beta$. Let f be differentiable at z_0 and$f'(z_0) \neq 0$.

Since both z and f are differentiable at t_0 and $z(t_0)$, respectively, $w = f \circ z$ is differentiable at t_0. Thus the image curve inherits its first-order behavior from both the geometry of the original curve and the local action of the mapping. Hence

$$w'(t_0) = f'(z_0)z'(t_0). \tag{3}$$

Now $w'(t_0) \neq 0$ since $f'(z_0) \neq 0$ by assumption and $z'(t_0) \neq 0$. Therefore the curve $\Gamma : w(t)$ through $w_0 = f(z_0)$ has a tangent ray $T_\Gamma(w_0)$. So, Equation (3) applied to each of two curves C_1, C_2 in $\mathbb{C}$, gives

$$\angle(\Gamma_1, \Gamma_2) = Arg\frac{z_2'(t_0)f'(z_0)}{z_1'(t_0)f'(z_0)} = Arg\frac{z_2'(t_0)}{z_1'(t_0)} = \angle(C_1, C_2).$$

Since the angle from C_1 to C_2 is defined as the angle from T_1 to T_2, and since $\angle(\Gamma_1, \Gamma_2) = \angle(C_1, C_2)$, *the mapping by an analytic function f with $f'(z_0) \neq 0$ preserves 'sense of rotation' as well as magnitude of angles* *at* z_0. A mapping having these properties is called conformal at z_0. Thus conformality is precisely the preservation of first-order angular geometry.

Conceptual conclusion. Conformality along a curve requires two independent nondegeneracy conditions: the curve must possess a well-defined first-order direction ($z'(t_0) \neq 0$), and the mapping must act locally as a nondegenerate complex scaling ($f'(z(t_0)) \neq 0$). When either condition fails, the first-order directional structure required to define angles collapses and conformality is lost.

7 Local mapping by an analytic function

From the definition of the derivative, we have

$$lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)|.$$

So, if $|z - z_0|$ is small, $|f(z) - f(z_0)|$ approximates $|z - z_0||f'(z_0)|$. From this and conformality we see that if $f'(z_0) \neq 0$, then a 'sufficiently small' triangle with vertex at z_0 is mapped into a geometrically similar 'curvilinear' triangle, and the lengths of the sides of the triangle are approximately $|f'(z_0)|$ as long as the corresponding sides of the triangle that is mapped from. Since $w'(t_0) = f'(z_0)z'(t_0)$ implies $arg\ w'(t_0) = arg\ z'(t_0) + arg\ f'(z_0)$, the mapping is a rotation through angle $arg\ f'(z_0)$ combined with an isotropic (same in all directions) stretching in the ratio $|f'(z_0)| : 1$. Geometrically, infinitesimal figures are rotated and uniformly scaled, but not sheared or distorted in angle. Thus, it behaves locally like the simple mapping $w = az$ near $z = 0$, where $a = f'(z_0)$. However, there is a difference. For the mapping $w = az$, the rotation (through angle $Arg\ a$) and stretching (by $|a|$) are the same throughout $\mathbb{C}$, not merely locally.

8 Geometric meaning of $z'(t_0) \neq 0$ condition.

The requirement $z'(t_0) \neq 0$ ensures that the path possesses a well-defined first-order direction at t_0. Indeed, writing

$$z(t) = x(t) + iy(t),$$

we have

$$z'(t_0) = x'(t_0) + iy'(t_0).$$

When $z'(t_0) \neq 0$, the first-order expansion

$$z(t_0 + h) = z(t_0) + h\, z'(t_0) + o(h)$$

shows that, to first order, the curve near t_0 is a straight segment in the direction $z'(t_0)$. This direction determines the tangent line, the limiting direction of secants, and the local angle structure required for conformality.

What if $z'(t_0) = 0$? If $z'(t_0) = 0$, then both $x'(t_0)$ and $y'(t_0)$ vanish, and the first-order term disappears:

$$z(t_0 + h) = z(t_0) + o(h).$$

Thus the curve has no first-order direction at t_0; the point becomes geometrically degenerate. This is a stronger phenomenon than in real calculus: for a real graph $y = f(x)$, the condition $f'(x_0) = 0$ still yields a horizontal tangent, whereas $z'(t_0) = 0$ yields no first-order direction at all.

Higher-order behavior. Let $m \geq 2$ be the smallest index such that $z^{(m)}(t_0) \neq 0$. Then

$$z(t_0 + h) = z(t_0) + \frac{h^m}{m!}\, z^{(m)}(t_0) + o(h^m).$$

The local geometry is then governed by this higher-order term. If m is odd, the curve crosses its limiting direction; if m is even, the curve touches and turns back. In either case, the first-order direction is absent.

9 Functions with nonzero derivative.

We have seen above that a function f with nonzero derivative at a point z_0 is conformal on C^1 curves for which $z'(t_0) \neq 0$.

Theorem 4. *Let f be a complex function continuous in a neighborhood of $z(t_0)$, and let*

$$z : [\alpha, \beta] \to \mathbb{C}$$

be a C^1 regular curve. Then f is conformal at the point $z(t_0)$ (along the curve) if and only if

$$f'(z(t_0)) \neq 0 \quad \textit{and} \quad z'(t_0) \neq 0.$$

Proof. Assume first that $f'(z(t_0)) \neq 0$ and $z'(t_0) \neq 0$. Since the curve is regular at t_0, it possesses a well-defined first-order direction given by $z'(t_0)$. The differentiability of f at $z(t_0)$ yields the local expansion

$$f(z(t_0+h)) = f(z(t_0)) + f'(z(t_0))\big(z(t_0+h) - z(t_0)\big) + o\big(|z(t_0+h) - z(t_0)|\big).$$

Thus, to first order, f acts as multiplication by the nonzero complex number $f'(z(t_0))$, which preserves angles and orientation. Hence f is conformal at $z(t_0)$ along the curve.

Conversely, suppose f is conformal at $z(t_0)$ along the curve. Conformality requires preservation of the angle between secant directions approaching t_0. If $z'(t_0) = 0$, the curve loses its first-order direction and secant directions need not converge uniquely, so conformality cannot hold. Thus $z'(t_0) \neq 0$.

Similarly, if $f'(z(t_0)) = 0$, then the first-order term in the expansion of f vanishes and f locally collapses directions, destroying the angle structure. Hence $f'(z(t_0)) \neq 0$.

Therefore both conditions are necessary, completing the proof. □

Remark 5 (Analytic functions with nonzero derivative)**.** *If f is analytic and $f'(z_0) = 0$, then f is not conformal at z_0. In this case the first-order term in the local expansion vanishes, and the behavior of f is governed by the first nonzero higher derivative. More precisely, if $m \geq 2$ is the smallest index such that $f^{(m)}(z_0) \neq 0$, then*

$$f(z) = f(z_0) + \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m + o\big(|z - z_0|^m\big).$$

Geometrically, the mapping locally behaves like $z \mapsto z^m$ near z_0: directions are multiplied m-fold and angles are not preserved. Such a point is called a critical point *(or branch point when $m \geq 2$). The failure of conformality here is thus another manifestation of first-order degeneration.*

Some analytic functions are not globally injective (for example e^z). By considering the range of such a function as a Riemann surface, we get inverse of the function; the domain of the inverse being the Riemann surface. Instead of extending the range of f to a Riemann surface, we may restrict the domain of f to sufficiently small neighborhood of z_0, provided that $f'(z_0) \neq 0$. Then in this neighborhood the function has a local inverse.

Theorem 6 (Local inverse)**.** *Analytic function f has a local inverse at z_0 if and only if $f'(z_0) \neq 0$.*

It is sufficient to show that f is injective in a sufficiently small neighborhood of z_0. We omit the proof which is in most textbooks on complex analysis.

Remark 7 (Derivative of the local inverse)**.** *Let f be holomorphic in a neighborhood of z_0 with $f'(z_0) \neq 0$, and set $w_0 = f(z_0)$. By Theorem 5, f admits a local inverse f^{-1} defined in a neighborhood of w_0. Differentiating the identity*

$$f^{-1}(f(z)) = z$$

and applying the chain rule at $z = z_0$ gives

$$(f^{-1})'(w_0)\, f'(z_0) = 1,$$

hence

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)}.$$

10 Globally conformal functions

Remark 8 (Functions with nonzero derivative everywhere)**.** *Some fundamental analytic mappings have nonzero derivative at every point and are therefore conformal wherever defined. Two basic examples are:*

- *The exponential function $f(z) = e^z$, for which*

  $$f'(z) = e^z \neq 0 \quad \text{for all } z \in \mathbb{C}.$$

  Thus e^z is conformal everywhere in the complex plane.

- *The Möbius (fractional linear) transformation*

  $$T(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0,$$

  for which

  $$T'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$$

  at every point where T is defined. Hence every Möbius transformation is conformal on its domain (the extended complex plane minus the pole $z = -d/c$ when $c \neq 0$).

These examples illustrate the global version of the local principle established above: a mapping is conformal precisely where its derivative does not vanish.